11.4. A SIMILARITY PROBLEM FOR TOEPLITZ OPERATORS*

Consider the Toeplitz operator T_F acting on $H^2 = H^2(\mathbb{D})$, where F is a rational function, with $F(\mathbb{T})$ contained in a simple closed curve Γ. Let τ be the conformal map from $\mathbb{D}$ to the interior of Γ, and say that F *backs up* at $e^{i\theta}$ if $\arg \tau^{-1}F(e^{it})$ is decreasing in some closed interval $[\theta_1, \theta_2]$, where $\theta_1 < \theta_2$ and $\theta_1 \leqslant \theta \leqslant \theta_2$. Let $\gamma_1, \ldots, \gamma_n$ be disjoint arcs on $\mathbb{T}$ such that F is one-to-one on each γ_i and such that $\bigcup \gamma_i$ is the set of all points where F backs up. Several recent results suggest the following conjecture.

Conjecture. Suppose $F(e^{it})$ has winding number $\nu \geqslant 0$. Then T_F is similar to

$$T_{\tau(z^\nu)} \oplus M_1 \oplus \ldots \oplus M_n \tag{1}$$

if $\nu > 0$, and to

$$M_1 \oplus \ldots \oplus M_n \tag{2}$$

if $\nu = 0$, where M_i is the operator of multiplication by $F(e^{it})$ on $L^2(\gamma_i)$.

One case of the above conjecture goes back to Duren [1], where it was proved for $F(z) = \alpha z + (\beta/z)$, $|\alpha| > |\beta|$. In this case $\nu = 1$ and F never backs up, so that $M_1, \ldots, M_n$ are not present in (1). Actually, Duren did not obtain similarity, but proved that T_F satisfies

$$LT_F = T_g L,$$

where L is some conjugate-linear operator $(L(\lambda_1 x + \lambda_2 y) = \bar{\lambda}_1 L(x) + \bar{\lambda}_2 L(y))$ and g is the mapping function for the interior of $\operatorname{clos} F(\mathbb{T})$. In [2], the conjecture was proved in case F is ν-to-one in some annulus $s \leqslant |z| \leqslant 1$. Here again F never backs up. In [3], F was assumed to have the form

$$F = \varphi/\psi, \tag{3}$$

where φ and ψ are finite Blaschke products, ψ having only one zero. In this case $\tau(z) = z$ and n can be taken to be 1.

The main tool used in [3] was the Sz.-Nagy–Foiaş characteristic function of T_F, which we computed explicitly and which, as we showed, has a left inverse. A theorem of Sz.-Nagy–Foiaş [4, Theorem 1.4], was then used to infer similarity of T_F with an isometry. Moreover, the unitary part in the Wold Decomposition of the isometry could be seen to have multiplicity 1, and so the proofs of the representations (1) and (2) were reduced to spectral theory.

If F is of the form (3) where φ and ψ are finite Blaschke products, ψ having *more than one* zero, the computation of the characteristic function of T_F is no longer easy. However, left invertibility can sometimes be proved without explicit computation. This is the case if φ and ψ have the same number of zeros, i.e., when T_F is similar to a unitary operator. This and some other results related to the conjecture are given in [5]. The Sz.-Nagy–Foiaş theory may also be helpful in attempts to formulate and prove a version of the conjecture when (3) holds with φ and ψ arbitrary inner functions. For example, it follows from [3] that if φ is inner and ψ is a Blaschke factor, then T_F is similar to an isometry.

For the case in which Γ is not the unit circle, the only successful techniques so far are those of [2], which do not use model theory. We have not been successful in extending them beyond the case of F satisfying the "annulus hypothesis" described above. There is a model theory which applies to domains other than $\mathbb{D}$ [6], but to our knowledge no results on similarity are a part of this theory. More seriously, to apply the theory, one would need to know that the spectrum of T_F is a spectral set for T_F, a result which does not seem to be known for rational F at this time.

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Finally, it seems hardly necessary to give reasons why the conjecture would be a desirable one to prove. Certainly detailed information on invariant subspaces, commutant, cyclic vectors, and functional calculus would follow from this type of result.

LITERATURE CITED

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